

A MORE SENSITIVE LORENTZIAN STATE SUM

Louis Crane and David Yetter
Dept. of Mathematics
Kansas State University
Manhattan, KS 66503
crane@math.ksu.edu, dyetter@math.ksu.edu

ABSTRACT *We give the construction modulo normalization of a new state sum model for lorentzian quantum general relativity, using the construction of Dirac’s expanders to include quantum operators corresponding to edge lengths as well as the quantum bivectors of the Barrett Crane model, and discuss the problem of its normalization. The new model gives rise to a new picture of quantum geometry in which lengths come in a discrete spectrum, while areas have a continuum of values.*

I. INTRODUCTION

The state sum models for quantum general relativity in [1,2] have attracted considerable attention, especially since the discovery of their finiteness in [3]. Unfortunately, they seem to have two weaknesses. The first is the well noted difficulty of finding a classical limit for them, the second is the fact that they contain an excessively large contribution from degenerate geometries [4,5,6].

In hindsight, it is clear why such problems should arise. The models do not have quantum operators corresponding to edge lengths or components of edge vectors. In the classical derivation of the model, the constraints on the bivectors implied that edge lengths would exist, except in degenerate configurations. This led some researchers to hope that it would be possible to have “edges without edges,” i.e. to recover the geometry of the edges as a side effect of the constraints. The discovery of the degenerate configurations seems to indicate that this hope was misplaced.

The problem now arises of how to modify the model in such a way as to retain its positive features, while removing the degenerate configurations, and being able to impose more stringent matching conditions between adjacent 4-simplices, thus providing a more natural classical limit.

One promising approach to this problem was proposed in [6]. This approach may be described as conservative, in the sense that it retains the basic form of the model, as a multiple integral in hyperbolic space, while projecting out the dangerous region of the integral.

We want to propose a “radical” approach. We construct a state sum model which has both edge length and bivector operators. The new model will use the expanders of Dirac [7], as quantizations of the edges of a triangulation, decompose their tensor products to obtain unitary Lorentz representations to

represent quantizations of the bivectors on the faces as in the model of [2], then combine them in 10J symbols as in the BC model. The objective is to retain the dynamics of the BC model, while enlarging the variables to allow more precise joining conditions between simplices.

The expanders of Dirac can be thought of as a relativistic analog of the solutions of the spherical harmonic oscillator. Much as the eigenvectors of the spherical harmonic oscillator form a basis for $L^2(R^3)$, the expanders give a basis for $L^2(R^4)$, and can be considered a quantization of an edge displacement in a four dimensional simplex.

Very roughly, the physical idea is to augment the model by including solutions of the Schrodinger equation with radial terms analogous to the solutions for the hydrogen atom instead of just the angular pieces which correspond to spins (the harmonic oscillator in 3D as a radial potential, is very similar to, though not identical with, the hydrogen atom since they are both radial potentials). This is consonant with the intuition that the triangulations of this model are “atoms of geometry”. Motivated by this intuition, we will explore using the oscillator hamiltonian to normalize the theory.

Another intuitive way of understanding the new model is that the models of [1,2], were quantizations of a description of the geometry of a triangulated manifold by the bivectors on its 2-simplices, which can be related to the infinitesimal generators of the rotation group. The new model is a quantization of a description of the displacement vectors on edges and bivectors on 2-simplices, which can be related to the full set of generators of the poicare group, translations on edges, rotations on faces.

Our proposal can also be understood in a categorical context. The various state sum models for quantum GR are all constrained versions of categorical state sums related to topology [8]. From the higher categorical standpoint [9,10], it is rather odd that we are using only a tensor category, rather than a tensor 2-category in a 4D state sum. Indeed, [11,12] has shown that a state sum in 4D similar to the ones in [8], can be constructed from a “spherical” 2-category. The most obvious difference between the two types of state sums is that in the 2-categorical approach both edges and faces get information.

The new state sum we are proposing is related to a new 2-category [13] which corresponds to the combination of the poicare and lorentz groups into a categorical group [14,15]. The construction makes use of a new class of categorical structures called “measured categories.” Indeed, it is the coincidence of a natural higher categorical picture with some natural quantum geometry which motivates the model we will be describing. The categorical aspects of this construction will appear in a companion paper [13], the present paper, intended for an audience in mathematical physics, will not discuss them.

Since the expanders are no longer familiar, we will begin with an exposition of them, which should be self contained for anybody familiar with the unitary lorentz representations which play a role in [2]. Then we will outline the construction of the new model.

II. THE EXPANSORS

In 1945, Dirac [7] introduced the first study of unitary representations of noncompact lie groups by constructing a family of examples, which he called the expanders, for the lorentz group.

His construction is a lorentz signature analog of the three dimensional spherical harmonic oscillator. Let us first review the construction in the three dimensional euclidean signature case. We can take as representing spaces the homogeneous polynomials of degree n in the formal variables ξ_x, ξ_y, ξ_z .

The action on this space of the three dimensional rotation group $SO(3)$ is given by substitution of the rows of the matrices for group elements into the polynomials. A simple calculation shows that such a substitution preserves the inner product on the space of polynomials in which monomials with different exponents are orthogonal and each monomial $\xi_x^n \xi_y^l \xi_z^m$ has length $(n!l!m!)^{1/2}$. Thus the expanders of each degree form a unitary representation of $SO(3)$.

These representations are not irreducible. In general, the homogeneous polynomials of degree n decompose into a direct sum of one copy each of every irreducible representation whose parity is the same as n from 0 or 1 up to n . This fact is well known in nuclear physics. This result is closely analogous to the well known solution of the hydrogen atom, whose three quantum numbers would correspond to the degree n , the spin of the representation, and the z component of the spin.

The relationship between the representations so constructed, which Dirac called expanders because they come from an expansion of a polynomial under a substitution, and the solutions of the spherical harmonic oscillator, is that the ξ variables correspond to the raising operators for the three harmonic oscillators in the three dimensions.

$$\xi_x = x + d/dx \text{ etc.}$$

The inner product given on the basis above then corresponds to the ordinary inner product in $L^2(R^3)$ as a simple calculation shows. The degree of the polynomial is just the energy level of the solution.

This construction is not original to Dirac. His contribution was extending the construction to the lorentzian signature, while preserving the unitary nature of the representations.

Before we discuss the lorentzian signature result, we will briefly describe the four dimensional analog of this construction, because it may well prove interesting in the study of euclidean quantum gravity.

The construction in 4D is the same as in 3D except we must include a fourth ξ variable. In complete analogy to 3D; we thus obtain for each n a reducible unitary representation of $SO(4)$.

The decomposition into irreducibles is similar, with the interesting new feature that only the balanced representations of $SO(4)$ now appear. For the case

of $SO(4)$, this means that the two half integers indexing the representation are equal. This is not surprising, since only they appear as harmonics on S^3 . Nevertheless, for readers familiar with the development to date in the state sum models, it is already clear that certain simplifications are going to ensue.

In order to use a similar construction to obtain unitary representations of $SO(3,1)$, Dirac hit on the expedient of including a fourth parameter ξ_t , and considering the homogeneous polynomials in ξ_x, ξ_y, ξ_z , and $(\xi_t)^{-1}$. This is no longer a finite dimensional space for any positive or negative degree, as is necessary, since $SO(3,1)$ is noncompact. Assigning a length

$$(l!m!n!)^{1/2}(p!)^{-1/2}$$

to the monomial

$$\xi_x^l \xi_y^m \xi_x^n \xi_t^{-p}$$

we again find that $SO(3,1)$ preserves the inner product on the space of homogeneous expanders of degree n , which we denote E^n .

This discovery of Dirac's was the first example of a family of unitary representations of a noncompact group to appear. Just as in the Euclidean case, the representations compose into series of irreducibles.

The irreducibles which appear are in the principal series [16,17] of representations $R(n, \rho)$ of $SO(3,1)$ for n integer and ρ real. Interestingly, only the balanced representations $R(n,0)$ and $R(0,k)$ appear in them. Negative degree extensors decompose as strings of $[n/2]$ $R(i,0)$'s of the same parity as n , followed by all the $R(0,i)$'s, while the positive degree extensors decompose into strings of $R(0,i)$'s only, beginning with $i=n$ and containing all the $R(0,i)$'s of appropriate parity. These facts are easy to check, comparing the formulas in [7] and [16].

The expanders have an interpretation in terms of the tensor product of 4 harmonic oscillators with the oscillator in the t direction assigned a negative energy. The degree is then the energy level of the solution.

Using the eigenbasis for the "lorentzian harmonic oscillator", we can thus quantize minkowski space in the form

$$L^2(M^4) = \oplus E^n;$$

with the further decomposition as representations of $SO(3,1)$ following immediately.

The positive and negative parts of our sum quantize the timelike and space-like regions. We believe this could be treated using geometric quantization.

III. THE QUANTUM TRIANGLE

Now we have a tool for constructing a quantized description of the geometry on a lorentzian simplex. We first need to see how to combine data on a triangle to produce quantum edges and a quantum bivector.

The classical conditions we need to impose are that the vector sum of the displacements on the three edges is 0, and that the bivector is the wedge product of any two of them, with a sign depending on orientation. It is actually slightly easier to order the 3 vertices of the triangle 1,2,3, orient the edges by ascending order, and require the vector on the edge labelled 13 to be the sum of the other two vectors.

It is easy to see how to quantize this. The space $T_q = (\oplus E^n)^{\otimes 2}$ is naturally identified with the space $T_S = L^2[\{(a,b,c) \mid a,b,c \in M^4; c = b + a\}]$. We can restrict to the skew symmetric part of T_q . Passage from one ordering to another would give us a different description of the same space. Explicitly,

$$\sigma_q; L^2[a,b] \leftrightarrow L^2[a,c] \text{ for } c = b + a.$$

Furthermore, this map from one parametrization of T_S to another is equivariant with respect to the action of $SO(3,1)$, so the decomposition of the skew part of T_q would still give the same combination of irreducible representations of $SO(3,1)$ after a change of ordering. Thus, the sum of representations of $SO(3,1)$ occurring in T_q^A is a natural setting for a quantization of the bivector in the quantum triangle. The action of the operators corresponding to components of the bivector is just the action of the lie algebra elements on the representations, as in the correspondence in [1,2] between the lie algebra and the bivectors.

We now have a space, T_S identified with its presentation T_q , which is a quantization of a minkowskian triangle, while its skew symmetric part is a natural setting for the quantum bivector.

How should we extract a “quantum bivector?” To see this, we should think about the geometry of the situation where two vectors are wedged into a bivector. The map

$$\pi : v_1 \times v_2 \rightarrow v_1 \wedge v_2$$

is a fibration with fiber $R^2 \times S^1$, since the set of vectors with a given wedge is the set of pairs in the plane which make a parallelogram with a given area. The image of π is the set of simple bivectors of M^4 .

We want a quantization of the image of π , so we want to identify the representations in the decomposition of T_q^A which pass down to the image. The answer is interesting, if familiar. The S^1 piece of the fiber of π is rotated by the rotation in the plane of the bivector to which it is sent. Thus, the part of the function space on the fiber which is constant along S^1 transforms the same way as the corresponding function on the image.

Now, which representations correspond to constants along S^1 , and which to higher Fourier components? The little group of a bivector is contained in

a maximal torus of $SO(3,1)$, so the representations of the principal series are induced from it. Tensoring with a nontrivial fourier component will therefore change a representation $R(0, \rho)$ to $R(k, \rho)$.

Thus, in order to make a quantum description of the bivector on the quantum triangle, we must project out the unbalanced representations appearing in T_q^A .

This result is not surprising, since we should only obtain simple bivectors when we wedge two vectors, but it is reassuring that it emerges naturally from our categorical quantization procedure. We will now have an infinite overcounting of the representations which survive in the image of π , from the tensor product with the $L^2(R^2)$ piece from the fiber of π . It is already clear from the definition of T_q that it will contain infinitely many copies of each representation.

Let us describe more explicitly the representation of the quantum bivector in the quantum triangle. We are taking the skew part of the tensor product of the direct sum of all the expanders E^n . Each of these decomposes into a tower of copies of $R(0, \rho_i)$ for all integer valued ρ beginning with n and of the right parity. We then project onto the balanced part, ie. only the copies of $R(0, \rho)$, where now ρ is any positive real number. Thus we get copies of the direct integral of all the $R(0, \rho)$ for each combination of two indices n , and two ρ_i , skew symmetrized with respect to the pair of indices. The n indices are the quantum version of the length operators on the sides. The other index ρ_i is analogous to the angular momentum quantum number for the hydrogen atom. We do not yet understand its quantum geometry.

In this construction, we are using two different operations on the expensor space. One is the identification σ_q of $L^2[a, b]$ with $L^2[a, c]$ above, the other is the skew symmetric tensor product of representations of $SO(3,1)$. These are quantum versions of vector addition and the wedge product of vectors. The fact that our expensor space can both be naturally added and wedged, which we are exploiting here, is related to the categorical idea mentioned above, that representations of the poincare and lorentz groups should be fused into a higher categorical structure. T_q and its transformation σ_q are the same space and transformation which occur in the 2-categorical picture, where the natural decomposition of $L^2(M^4)$ is as a direct integral of characters of the translation group, and σ_q is simply tensor product of characters [13, 14, 15]. This view makes even clearer the fact that T_q is a good quantization of the triangle: tensor product of characters of the translation group is simply vector addition.

IV. AN OUTLINE OF THE NEW MODEL

Now that we have our basic toolkit, it is easy to see how to set about constructing a new model. We attach one copy of T_q to each triangle in the triangulation, decompose the alternating part of each into irreducible representations of $SO(3,1)$ by tensoring the decompositions of each E^n , project out all except the

balanced representations, form regularized 10J symbols from them as in [2], and form the sum over all choices of irreps in the triangulation of the product of all 10J symbols. We restrict our sum to terms in which all edges in four simplices which are incident in the triangulation have the same E^n . This is a quantization of requiring all matching edges to have the same lengths, all triangles to close, and the bivectors on faces to correspond to the edges.

At the price of vastly increasing our degrees of freedom, we now have a state sum with quantum edge variables which match between adjacent 4-simplices, and a dynamics which will reproduce the Einstein-Hilbert action by the usual argument.

We conjecture that the degenerate states of the old model [6] will now be of measure 0, since they do not correspond to any assignment of edge lengths. This is a rather subtle conjecture, since some global constraints must appear among the quantum bivector data on a 4-simplex.

Of course, in this naive unnormalized version of the model, the sum will be infinite. We discuss issues of normalization below.

It is interesting that the new model is a more complicated sum of combinations of balanced representations only, with more projections of tensor products of balanced representations onto balanced representations. This means it is built out of the same mathematical pieces as the B-C model [2], and can therefore be represented as multiple integrals of Feynman type over H^3 [18, 19]. This is an optimistic sign that it will have a good normalization.

To see more explicitly what the new model will look like, it is a good idea to pick an ordering of the vertices of a four simplex. We can then choose to represent the copies of T_S on the six triangles incident on the vertex labelled 1 by the copy of T_q corresponding to the two edges incident on 1, in lexicographic order. This means starting with 4 copies of $\oplus E^n$, one on each of the four edges out of vertex 1, decomposing into representations of $SO(3,1)$, and forming alternating tensor products in pairs. This can be thought of as a “quantum frame”. The other four triangles can be represented in terms of the representations on the four edges by using the transformation σ_q of T_S corresponding to a reordering. In further developing the model it will be necessary to give an explicit form for σ_q . It is analogous to the problem of writing a plane wave in terms of atomic orbitals, so familiar from scattering theory.

This procedure gives a map

$$\mu : (\oplus E^n)^{\otimes 4} \rightarrow \oplus R(0, \rho)^{\otimes 10}$$

which will need to be studied carefully to test the above conjecture.

V. ISSUES OF NORMALIZATION; SMALL ATOMS

Our first suggestion in regard to normalization relates to a change in point of view as to the meaning of terms in our quantum state sum.

We want to think of the quantum data on our 4-simplices as “atoms of geometry,” [20] rather than as possibly large regions which happen to be utterly flat. We have several reasons for this. In the first place, the idea of spacetime as a nondenumerably infinite point set seems inextricably wedded to classical physics. We believe the point of our spin foam models is to replace this picture with a superposition of discrete structures. In the second place, a large volume which is treated as completely flat is unphysical. We would also mention the unpleasant behavior of spin foam models in the large J limit as something to be avoided.

Motivated by the above considerations and by the connection between the decomposition of $L^2(R^4)$ we are using and the spherical harmonic oscillator hamiltonian, we propose to put another factor in our state sum. This would be a product of e^{-H} for each edge in the triangulation. Actually 2 choices are possible, we can either make the edges spacelike, or make them timelike and use the opposite sign convention in our definition of $H = x^2 + y^2 + z^2 - t^2$. In the expensor picture, this amounts to an exponentially damped weight on E^n terms.

Certainly, this would help a lot with the divergence. More delicate computation will be needed to see if it suffices.

It seems clear that the problem of the multiplicity of copies of $R(0, \rho)$ due to the indices n would be controlled by this ansatz. The other multiplicity needs to be better understood. It appears to be an artifact caused by the overcounting from the extra degrees of freedom in the fibration discussed above, but a nice way to cancel it is needed.

We ought to remind ourselves that the initial normalization of the BC model, motivated by the connection with TQFT, turned out to be not the best one, and that it was a rederivation of the model as an expansion in feynman diagrams that gave the correct one [21].

This poses the question of finding some feynmanological picture for the new model. We believe this is an interesting question for several reasons.

In the first place, the integrals in the BC model turned out to be essentially feynman integrals in curved space. The new model will be composed of slightly more complicated integrals of a similar type.

Secondly, feynman diagrams are essentially morphisms in a tensor category. The new model is 2-categorical in a geometrically natural way, so it should be a good place to try to understand the 2-categorical analog of feynmanology, which should be important in advancing the spin foam picture.

VI CONCLUSIONS

Let us restate the construction of the model as far as we currently understand it. We begin by summing over an assignment of one E^n to each edge in the triangulation. We then put a normalization factor of e^{-n} on each edge, impose the projection onto T^q for each triangle of each 4-simplex, decompose the skew

tensor product on each triangle into irreps of the lorentz algebra, project onto the balanced irreps, form the B-C 10J symbols, and take the integral of products of terms. It is not clear yet how much our constraints help us with our naive infinities, or whether further normalization is necessary. We believe a deeper study of the new quantum geometry of this model will clarify this.

The fibration picture for the quantum triangle suggests that the naive divergences in this model just come from an infinite multiplicity corresponding to the volume of $R^2 \times S^1$. This makes us optimistic that a consistent regularization can be found, much as the divergences in the model of [2] were all controlled by dividing out a single infinite volume.

The hope is that the classical states (terms of stationary phase) in this sum will be discrete einstein metrics with all simplices very small.

We note that the idea of [22], that we need to divide by an infinite diffeomorphism volume factor to regularize a state sum, is formally similar to the current situation. We are not sure if there is a deeper connection.

Since there is some interest in the question of discrete versus continuous spectra for geometrical quantities in quantum general relativity, let us note that the length variables in our model are discrete because of the decomposition of the expensor spaces described above, but that the area spectrum will be continuous because the decomposition of the tensor product of the irreducible unitary representations takes the form of a direct integral, not a discrete sum.

It is also interesting to consider the possibility of a q-version of the present model. Since it is well known that a natural q-harmonic oscillator exists [23], it is easy to see how to go about constructing q-expansors. A q-version of the B-C model has been studied in [24]. It would be interesting to study the decomposition of the q-expansors, and to see if they led to a manifestly finite version of the 2-categorical model.

At this point, it is still rather speculative to be thinking about adding matter to this system. Nevertheless, we point out that we are only including a factor which exponentially damps the length on each edge in our preliminary proposal for a normalization. This is in contrast to state sum models for TQFTs in which rather subtle terms arise, which would not make sense for nonmanifold configurations. It is therefore possible to consider the suggestion of conical matter [25] in the model of this paper.

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